Supplementary examination 2019
M.Math. II — Algebraic Geometry
Each question carries 20 marks.

Group A
Answer any one question from group A

Question 1
Let V be a non-empty variety in $\mathbb{A}^n$, $\Gamma(V)$ be the coordinate ring. For $P \in V$ let $\mathcal{O}_P(V)$ be the set of rational functions on V that are defined at P.
(a) Show that $\Gamma(V) = \cap_{P \in V} \mathcal{O}_P(V)$.
(b) Show that $\mathcal{O}_P(V)$ is a noetherian local domain.

Question 2
Let W be a subvariety of an affine variety V and let $I_V(W)$ be the ideal of $\Gamma(V)$ corresponding to W.
(a) Show that every polynomial function on V restricts to a polynomial function on W.
(b) Show that the map from $\Gamma(V)$ to $\Gamma(W)$ defined in part (a) is a surjective homomorphism with kernel $I_V(W)$.

Group B
Answer any four questions from group B

Question 3
(a) Let F be a projective plane curve. Show that a point P is a multiple point of F if and only if $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$.
(b) Show that the curve $XY^4 + YZ^4 + XZ^4$ is irreducible. Find the multiple points and the multiplicities and tangent lines at the multiple points.

Question 4
(a) State Max Noether's fundamental theorem.
(b) Let C, C' be plane cubics, $C \bullet C' = \sum_{i=1}^{9} P_i$. Let Q be a conic and $Q \bullet C = \sum_{i=1}^{6} P_i$. Assume $P_1, \ldots, P_6$ are simple points on C. Show that P_7, P_8, P_9 lie on a straight line.

Question 5
Let U_i be the subset of $\mathbb{P}^n$ defined as $U_i = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n | x_i \neq 0\}$ and equipped with the topology induced from $\mathbb{P}^n$. Let $\varphi_i : \mathbb{A}^n \longrightarrow U_i$ be

defined as $\varphi_i(a_1, \ldots, a_n) = [a_1 : \ldots : a_{i-1} : 1 : a_i : \ldots : a_n]$.
(a) Show that φ_i is a homeomorphism.
(b) Show that a set $W \subset \mathbb{P}^n$ is closed if and only if $\varphi_i^{-1}(W)$ is closed in $\mathbb{A}^n$ for $i = 1, \ldots, n+1$.

Question 6
(a) Let X be a variety. Define dimension of X.
(b) Let V^* be the projective closure of an affine variety V. Show that $dim\ V = dim\ V^*$.
(c) Show that $dim\ \mathbb{A}^n = dim\ \mathbb{P}^n = n$.

Question 7
(a) Show that two varieties are birationally equivalent if and only if their function fields are isomorphic.
(b) Show that every n-dimensional variety is birationally equivalent to a hypersurface in $\mathbb{A}^{n+1}$ or $\mathbb{P}^{n+1}$.

Question 8
Let C be an irreducible projective curve, $f : X \longrightarrow C$ be the birational morphism from the nonsingular model X onto C and $K = k(C) = k(X)$ be the function field.
(a) For $z \in K$, define $div(z)$, divisor of z.
(b) Show that for any $z \in K$, $div(z)$ is a divisor of degree zero.
(c) Let us assume that the above curve C is a plane curve of degree n. Let G be a plane curve of degree m and do not contain C as a component. Define $div(G)$, divisor of G and show that $div(G)$ is a divisor of degree mn.